Lecture 27 (11/29/21)

Recall. If f has an isolated sing at $z=a$, i.e. f is analytic in $B(a,r)\setminus\{a\}$, then it has a Laurent series expansion

$$f(z)=\sum_{n=-\infty}^{\infty} a_n (z-a)^n,$$

where $$a_n=\frac{1}{2\pi i}\oint_{|z-a|=\rho<r}\frac{f(z)}{(z-a)^{n+1}}\,dz.$$

In particular, $$a_{-1}=\frac{1}{2\pi i}\oint_{|z-a|=\rho} f\,dz.$$

Def. ① The Laurent series coefficient a_{-1} is called residue of f at $z=a$, denoted $\operatorname{Res}_{z=a} f$.

Residue Thm. Let f be analytic in G except for isol. sing. at $z_1, z_2, \ldots$ in G. Let γ be a closed curve s.t. $\gamma\approx 0$ in G. Then

$$\frac{1}{2\pi i}\int_\gamma f\,dz=\sum_{k=1}^{\infty}\operatorname{Res}_{z=z_k} f\cdot n(\gamma,z_k)$$

Rem. As in Arg. Princ., if there are ∞ many isol. sing. they must accumulate on ∂G and only finitely many has $n(\gamma, z_n) \neq 0$.

Pf. Let $B(z_k, r_k)$ be small balls at z_k, s.t. $B(z_k, r_k) \subseteq G$ and they are pairwise disjoint.

Let γ_k be $|z-a| = \rho_k < r_k$. Then $n(\gamma_k, z) = 1$ in $|z-a| < \rho_k$ and $= 0$ outside. It follows that

$$n(\gamma, z) - \sum_{k=1}^{\infty} n(\gamma, z_k)\, n(\gamma_k, z) = 0$$

for $z \in G \setminus \{z_1, z_2, \dots\}$. By Cauchy's Thm

$$0 = \int_{\gamma - \sum n(\gamma, z_k)\gamma_k} f\,dz = \int_\gamma f\,dz - \sum_{k=1}^{\infty} n(\gamma, z_k) \int_{\gamma_k} f\,dz$$

$$\Rightarrow \frac{1}{2\pi i}\int_\gamma f\,dz = \sum_{k=1}^{m} n(\gamma, z_k)\int_{\gamma_k} f\,dz$$

$$= \sum_{k=1}^{m} n(\gamma, z_k)\,\operatorname*{Res}_{z=z_k} f. \qquad \blacksquare$$

Useful formula for computing Res.

<u>Prop1.</u> Let g be analytic in $B(a,r)$ and $f(z) = \frac{g(z)}{(z-a)^m}$. Then

$$\operatorname*{Res}_{z=a} f = \frac{1}{(m-1)!}\, g^{(m-1)}(a).$$

Pf. DIY.

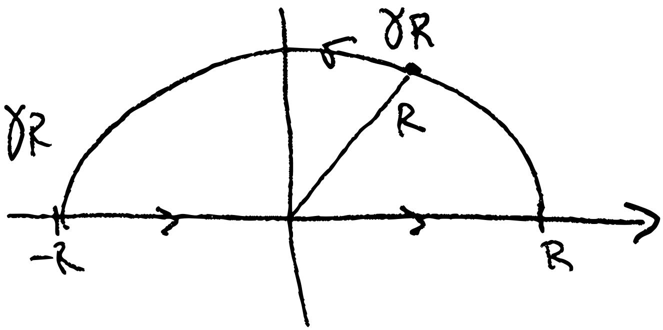
A typical application of the residue thm is calculation of integrals for which calculus techniques are not effective.

Ex ① Consider $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2}\,dx$.

Note that $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2}\,dx = \mathrm{Re}\int_{-\infty}^{\infty} \frac{e^{ix}}{1+x^2}\,dx$.

We fix $R >> 1$ and integrate $f(z) = \frac{e^{iz}}{1+z^2}$ over

$\gamma = [-R, R] + \gamma_R$



Since f has isolated sing. at $1+z^2=0$, i.e. $z = \pm i$ and only $z = +i$ is inside the curve we are integrating over, Res Thm $\Rightarrow$

$$\int_\gamma f\,dz = 2\pi i \operatorname*{Res}_{z=i} f = 2\pi i \cdot \left.\frac{e^{iz}}{z+i}\right|_{z=i} = \pi e^{-1}$$

$$\Rightarrow \int_{-R}^{R} \frac{e^{ix}}{1+x^2}\,dx + \int_{\gamma_R} \frac{e^{iz}}{1+z^2}\,dz = \frac{\pi}{e}$$

Let $R \to \infty$. We note $z = Re^{i\theta}$ on γ_R:

$$\left| \int_{\gamma_R} \frac{e^{iz}}{1+z^2} dz \right| \leq \int_{\gamma_R} \frac{|e^{iz}| \cdot |dz|}{|z|^2 - 1}$$

$$= \int_0^{\pi} \frac{e^{-R\sin\theta} \cdot R\, d\theta}{R^2 - 1} \leq \pi \frac{R}{R^2-1} \to 0$$

as $R \to \infty$. $\Rightarrow$

$$\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2} dx = \mathrm{Re} \int_{-\infty}^{\infty} \frac{e^{ix} dx}{1+x^2} = \frac{\pi}{e}.$$

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(2) Some further trickery allows us to compute

$$\int_0^{\infty} \frac{\sin x}{x} dx = \lim_{R \to \infty} \int_0^{R} \frac{\sin x}{x} dx$$

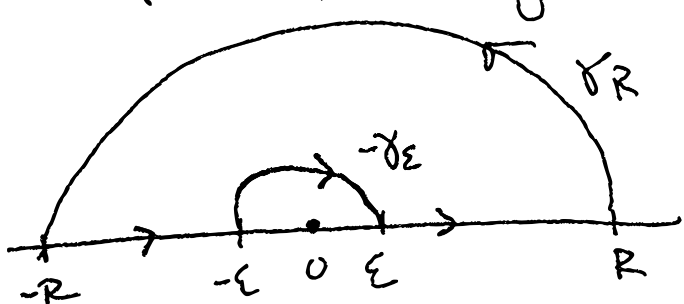
Note that this is not integrable in the usual sense.

The previous trick to consider $\frac{\sin x}{x} = \operatorname{Im} \frac{e^{ix}}{x}$ has a problem since $\operatorname{Re} \frac{e^{ix}}{x} = \frac{\cos x}{x}$, which is not integrable at 0.

We note that $\frac{\sin x}{x}$ is cont. at 0 and an even function of x. $\frac{\cos x}{x}$ is odd, so we can comput

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{1}{i} \lim_{\substack{\varepsilon\to 0\\ R\to\infty}} \int_{-R}^{-\varepsilon} \frac{e^{ix}}{x}\,dx + \int_{\varepsilon}^{R} \frac{e^{ix}}{x}\,dx$$

Now, $f(z) = \frac{e^{iz}}{z}$ has simple pole at $z=0$ and no other sing. CT $\Rightarrow$



$$\int_{-R}^{-\varepsilon} \frac{e^{ix}}{x}\,dx + \int_{-\gamma_\varepsilon} \frac{e^{iz}}{z}\,dz + \int_{\varepsilon}^{R} \frac{e^{ix}}{x}\,dx + \int_{\gamma_R} \frac{e^{iz}}{z}\,dz = 0$$

Thus, we must comput $\lim\limits_{\varepsilon\to 0}\int_{-\gamma_\varepsilon} f\,dz$ and

$\lim\limits_{R\to\infty}\int_{\gamma_R} f\,dz$.

<u>First:</u>

$$\left|\int_{\gamma_R}\frac{e^{iz}}{z}\,dz\right| \le \int_{\gamma_R}\frac{|e^{iz}|\cdot|dz|}{|z|} =$$

$$\int_0^\pi e^{-R\sin\theta}\,d\theta$$

<u>Lemma</u>. $\lim\limits_{R\to\infty}\int_0^\pi e^{-R\sin\theta}\,d\theta = 0$.

<u>Pf</u>. Choose $\varepsilon>0$. On $[\varepsilon,\pi-\varepsilon]$, we have $\sin\theta > \sin\varepsilon$,

$$\Rightarrow \lim_{R\to\infty}\int_0^\pi e^{-R\sin\theta}\,d\theta \le \lim_{R\to\infty}\left(\int_0^\varepsilon + \int_{\pi-\varepsilon}^\pi\right)\underbrace{e^{-R\sin\theta}}_{\le 1}\,d\theta$$

$$+\lim_{R\to\infty}\int_\varepsilon^{\pi-\varepsilon}\underbrace{e^{-R\sin\theta}}_{\le e^{-R\sin\varepsilon}\to 0}\,d\theta \le 2\varepsilon$$

$\Rightarrow \lim_{R\to\infty} \int_0^{\pi} e^{-R\sin\theta}\, d\theta = 0$ since $\varepsilon > 0$ is arbitrary. (Lemma also follows directly from Lebesgue's Dominated Conv. Thm.). $\blacksquare$

Now, for $\int_{-\gamma_\varepsilon} \frac{e^{iz}}{z} dz$, we note that

$g(z) = \frac{e^{iz}-1}{z}$ has removable sing. at $z=0$, and $g(0) = i \Rightarrow |g(z)| \le 2$ for $|z| \le \varepsilon$ if $\varepsilon > 0$ is small enough.

Then,

$$\lim_{\varepsilon\to 0} \left| \int_{-\gamma_\varepsilon} g\, dz \right| \le \lim_{\varepsilon\to 0} -\int_0^{\pi} 2\cdot \varepsilon\, d\theta = 0$$

$$\Rightarrow \lim_{\varepsilon\to 0} \int_{-\gamma_\varepsilon} \frac{e^{iz}}{z} dz = \lim_{\varepsilon\to 0} \underbrace{\int_{-\gamma_\varepsilon} \frac{1}{z} dz}_{-\int_0^{\pi} \frac{i\varepsilon e^{i\theta}}{\varepsilon e^{i\theta}} d\theta}$$

$$= -i\pi$$

Now, putting it all together

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{1}{2i}\left(\lim_{\varepsilon \to 0} \int_{\gamma_\varepsilon} \frac{e^{iz}}{z}\,dz - \lim_{R \to \infty} \int_{\gamma_R} \frac{e^{iz}}{z}\,dz \right)$$

$$= \frac{\pi}{2}.$$